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Jacobian for spherical coords

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

$$J = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos^2\phi\, \rho^2 \sin\phi + \rho^2 \sin^2\phi \sin\phi = \rho^2 \sin\phi$$

$x^2+y^2+z^2 = \rho^2$

ex) Compute $\int_R (x^2+y^2+z^2)^2\, dV$ where R is the solid ball of radius 5 about the origin.

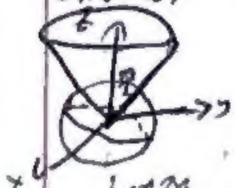
$R_{sph} = \{(\rho, \phi, \theta) : 0 \le \rho \le 5,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

$$\int_{\rho=0}^{5}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} (\rho^2)^2 \cdot \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

$$= \int_{\rho=0}^{5}\int_{\theta=0}^{2\pi} [-\cos\phi]_0^{\pi}\, d\theta\, d\rho = \int_{\rho=0}^{5} \rho^6 \int_{\theta=0}^{2\pi} 2\, d\theta\, d\rho = \int_{\rho=0}^{5} \rho^6 [4\pi]$$

$$= 4\pi \frac{(5)^7}{7}$$

2) $\iiint y^2 z\, dV$, R is the region above the cone w/ point at the origin and making an angle of $\frac{\pi}{3}$ radians w/ the positive z-axis and inside sphere w/ radius 2 centered at the origin



$0 \le \rho \le 2$
$0 \le \theta \le 2\pi$
$0 \le \phi \le \frac{\pi}{3}$

$$\rightarrow \int_{\rho=0}^{2}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/3} (\rho\sin\theta\sin\phi)^2(\rho\cos\phi) \cdot \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$$

$$= \int_0^2\int_0^{2\pi}\int_0^{\pi/3} \rho^5 \sin^2\theta \sin^3\phi \cos\phi\, d\phi\, d\theta\, d\rho = \frac{9}{64}\int_0^2\int_0^{2\pi} \rho^5 \sin^2\theta\, d\theta\, d\rho = \int_0^2 \frac{9}{128}\rho^5\, d\rho \int (1-\cos 2\theta)\, d\theta\, d\rho$$

$$= \frac{18\pi}{128}\int_0^2 \rho^5 = \frac{18\pi}{128} = \frac{9\pi}{64}\int_0^2 \rho^5 = \frac{9\pi}{64}\left(\frac{\rho^6}{6}\right)\Big|_0^2 = \frac{64}{6}\left(\frac{9\pi}{64}\right) = \frac{3\pi}{2}$$

ex) $\int_R 6xy\, dV$ $R=\{(x,y,z): 0\le y\le 1,\ y\le x\le 2y,\ 0\le z\le x+y\}$

$\int_{y=0}^{1}\int_{x=y}^{2y}\int_{z=0}^{x+y} 6xy\, dz\,dx\,dy \Rightarrow \int_{y=0}^{1}\int_{x=y}^{2y} 6xyz\Big|_0^{x+y} \Rightarrow \int_{y=0}^{1}\int_{x=y}^{2y} 6xy(x+y) = \iint_{x=y}^{2y} 6x^2y+6xy^2$

$=\int_{y=0}^{1} \frac{6}{3}x^3y+\frac{6}{2}x^2y^2\Big|_y^{2y} \Rightarrow 2(2y)^3y+3(2y)^2y^2-2y^3y-3y^2y^2 \Rightarrow 16y^4+12y^4-2y^4-3y^4$

$\Rightarrow 23\int_{y=0}^{1} y^4 \Rightarrow \frac{23}{5}y^5\Big|_0^1 = \frac{23}{5}$

2) $\iiint_R y\,dV \Rightarrow R:\{(x,y,z): 0\le x\le 3,\ 0\le y\le x,\ x-y\le z\le x+y\}$

$\int_{x=0}^{3}\int_{y=0}^{x}\int_{z=x-y}^{x+y} y\,dV \Rightarrow yz\Big|_{x-y}^{x+y} = y(x+y)-y(x-y) = yx+y^2-yx+y^2 \quad \int_{x=0}^{3}\int_{y=0}^{x} 2y^2$

$\Rightarrow \frac{2}{3}y^3\Big|_0^x = \int_{x=0}^{3}\frac{2}{3}x^3 \Rightarrow \frac{2}{12}x^4\Big|_0^3 = \frac{1}{6}\left(3^4\right) = \frac{27}{2}$

Cylindrical

1) $\iiint_R xy^2z\,dV \to R$ is bound by: $x=4y^2+4z^2$ and $x=4$

(x,r,θ) $\quad 4r^2\le x\le 4$, $0\le r\le 1$, $0\le\theta\le 2\pi$

$dA_{cart} = r\,dA_{cyl}$

$x=4y^2+4z^2 = 4(y^2+z^2) = 4r^2$

$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{x=4r^2}^{4} x(r\cos\theta)^2\, r\sin\theta\, r\,dx\,dr\,d\theta \Rightarrow \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 8(r^4-r^8)\cos^2(\theta)\sin(\theta)\,dr\,d\theta$

$=\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^4\cos^2\theta\sin\theta\,(8-8r^4) = \frac{8}{5}r^5\cos^2\theta\sin\theta - \frac{8}{9}r^9\Big|_0^1 = \frac{8}{5}\cos^2\theta\sin\theta - \frac{8}{9}\cos^2\theta\sin\theta$